The Strong Effects of Weak Externalities on School Choice

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Introduction

- Abdulkadiroğlu & Sönmez (2003) introduce in the literature of mechanism design the school choice problems, in which the goal is to match students with slots in public schools.

- In a school choice problem, each student has preferences for schools, which have a number of available seats and determine priority orders for students.

- One of the objectives of this literature is to characterise the existence of matchings—distributions of students in schools—that have good properties, such as stability against deviations by groups of students or efficiency in the distribution of seats in schools.

- Furthermore, the goal is to characterize the existence of mechanisms that allow for the implementation of stable and/or efficient matchings in situations where preferences are not observable.

Introduction

- In Abdulkadiroğlu & Sönmez (2003) model, each student is only concerned with ranking schools.
- This ensures that there are always stable matchings with good efficiency properties. Moreover, there are mechanisms that implement these matchings and give incentives to each student to report their true preferences, independent of what other applicants do.
- However, it is natural for students to consider the situation of other applicants when ranking schools. To capture this dimension it is necessary to introduce externalities into the model.
- In this type of problem, the presence of externalities compromises the existence of stable matches (Dutta & Massó (1997), Echenique & Yenmez (2007), Bykhovskaya (2020)).

In this talk....

- In school choice problems with externalities, we study students' incentives to reveal information about their school preferences.

- We assume that externalities are weak, in the sense that they are a secondary factor when evaluating a matching.

- We show that there are no mechanisms that are stable and strategy-proof or stable and (weakly) Pareto efficient.

- We determine restrictions on the priority orders of schools that ensure the existence of stable, strategy-proof and Pareto-efficient mechanisms.

Model

A *school choice problems with externalities*

$$(S, H, (\succ_s)_{s\in S}, (q_s)_{s\in S}, (R_h)_{h\in H})$$

is characterised by:

- S : Set of schools.
- H : Set of students.
- $\succ_s$: strict priority order of school s over students.
- q_s : quota of school s. We assume $|H| \leq \sum_{s\in S} q_s$.

Let $\mathcal{M}$ the set of *matchings*. That is, the set of functions, $\mu : H \to S \cup \{\otimes\}$ such that $|\mu^{-1}(s)| \leq q_s$, $\forall s \in S$.

- R_h : complete and transitive preference of student h defined over $\mathcal{M}$.

School choice context: $(S, H, \succ, q) \equiv (S, H, (\succ_s)_{s\in S}, (q_s)_{s\in S})$.

Model

A preference R_h defined on $\mathcal{M}$ is *egocentric* if, given $\mu, \eta \in \mathcal{M}$:

- If μ and η are indifferent under R_h, then $\mu(h) = \eta(h)$.
- If μ is strictly preferred to η and $\mu(h) \neq \eta(h)$, then $\mu' P_h \eta'$ when $\mu'(h) = \mu(h)$ and $\eta'(h) = \eta(h)$, where P_h is the strict part of R_h.

We will denote by $\mathcal{R}^{\text{ego}}$ the set of profiles $(R_h)_{h \in H}$ such that each preference relation R_h is egocentric.

An egocentric preference R_h naturally induces a *standard preference* $\sigma(R_h)$ defined on $S \cup \{\otimes\}$, which is complete, transitive and strict.

Given $R = (R_h)_{h \in H} \in \mathcal{R}^{\text{ego}}$, we will denote by $\sigma(R) \equiv (\sigma(R_h))_{h \in H}$ the profile of induced standar preferences.

We will denote by $\mathcal{R}^{\text{std}}$ the set of profiles $(\sigma_h)_{h \in H}$ such that each σ_h is a linear order defined over $S \cup \{\otimes\}$.

Stability

Given a matching $\mu : H \to S \cup \{\otimes\}$, we will say that:

- μ es *individually rational* if there does not exist $h \in H$ such that $\otimes\, \sigma(R_h)\, \mu(h)$.
- μ es *envy-free* if there does not exist a pair $(s, h) \in S \times H$ such that $|\mu^{-1}(s)| = q_s$ and for any $h' \in \mu^{-1}(s)$ we have that

$$h \succ_s h', \qquad s\, \sigma(R_h)\, \mu(h).$$

- μ *wasteful* if there exists $(s, h) \in S \times H$ such that

$$|\mu^{-1}(s)| < q_s, \qquad s\, \sigma(R_h)\, \mu(h).$$

A matching $\mu \in \mathcal{M}$ is stable if is individually rational, non-wasteful, and envy-free.

Efficiency

- A matching is *Pareto efficient* if there is no alternative way of distributing school places that improves the situation of at least one student without disadvantaging the others.

- A matching is *weakly Pareto efficient* if there is no alternative way of distributing school places that is preferred by all students.

Mechanisms

Let $\mathcal{R} = \prod_{h \in H} \mathcal{R}_h$, where $\mathcal{R}_h$ is a preference domain for $h \in H$.

- A *mechanism* is a function $\varphi : \mathcal{R} \to \mathcal{M}$ that selects a matching for each student preference profile.

- A mechanism φ es *stable* if for all $R \in \mathcal{R}$, $\varphi(R)$ is stable in $(S, H, \succ, q, R)$.
- A mechanism φ es *strategy-proof* if for each $h \in H$ we have that

$$\varphi(R)\ R_h\ \varphi(\widetilde{R}_h, R_{-h}), \qquad \forall R \in \mathcal{R}, \forall \widetilde{R}_h \in \mathcal{R}_h.$$

  When φ is *strategy-proof*, for each student it is a weakly dominant strategy to report his true preferences.

- A mechanism φ es *bossy* if there exists $h \in H$, $R \in \mathcal{R}$ and $\widetilde{R}_h \in \mathcal{R}_h$ such that

$$\varphi(\widetilde{R}_h, R_{-h})(h) = \varphi(R)(h) \quad \text{y} \quad \varphi(\widetilde{R}_h, R_{-h}) \neq \varphi(R).$$

  When φ is *bossy*, there are scenarios in which at least one student could report false preferences with the aim of altering the implemented matching without changing their assigned school.

Mechanisms

Deferred Acceptance with Egocentric Preferences

Consider the mechanism $AD^{\text{ego}}_{(\succ,q)} : \mathcal{R}^{ego} \to \mathcal{M}$ characterised by

$$AD^{\text{ego}}_{(\succ,q)}(R) = AD_{(\succ,q)}(\sigma(R)).$$

Top Trading Cycles with Egocentric Preferences

Consider the mechanism $TTC^{\text{ego}}(\succ, q) : \mathcal{R}^{ego} \to \mathcal{M}$ characterised by

$$TTC^{\text{ego}}_{(\succ,q)}(R) = TTC_{(\succ,q)}(\sigma(R)).$$

Incompatibility between stability and strategy-proofness

Theorem 1

There are contexts $(S, H, \succ, q)$ in which no stable and strategy-proof mechanism exists.

Proof. Assume that $S = \{s_1, \ldots, s_n\}$ and $H = \{h_1, \ldots, h_n\}$.

Each school has one quota available ($q_s = 1$, for all $s \in S$).

The priority structure $\succ = (\succ_s)_{s \in S}$ is characterised by

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\cdots$	$\succ_{s_{n-1}}$	$\succ_{s_n}$
h_2	h_1	h_3	h_4	$\cdots$	h_{n-1}	h_2
h_3	h_2	$\vdots$	$\vdots$	$\vdots$	$\vdots$	h_n
h_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Incompatibility between stability and strategy-proofness

Theorem 1

There are contexts $(S, H, \succ, q)$ in which no stable and strategy-proof mechanism exists.

*Proof (continued).*Let $R = (R_h)_{h \in H} \in \mathcal{R}^{\text{ego}}$ an egocentric preference profile such that $\sigma(R)$ satisfies the following properties:

$\sigma(R_{h_1})$	$\sigma(R_{h_2})$	$\sigma(R_{h_3})$	$\cdots$	$\sigma(R_{h_n})$
s_1	s_2	s_3	$\cdots$	s_n
s_2	s_1	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$

Given the matchings

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ s_1 & s_2 & s_3 & \cdots & s_n \end{pmatrix} \qquad \text{y} \qquad \mu' = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ s_2 & s_1 & s_3 & \cdots & s_n \end{pmatrix},$$

assume that the student's preferences h_3 satisfy $\mu' P_{h_3} \mu$.

Incompatibility between stability and strategy-proofness

Theorem 1

There are contexts $(S, H, \succ, q)$ in which no stable and strategy-proof mechanism exists.

Proof (continued).
In this context, μ and μ' are the **only** stable matchings of the school choice problem $(S, H, \succ, q, R)$. Therefore, if $\Gamma : \mathcal{R}^{\text{ego}} \to \mathcal{M}$ is a stable mechanism, then $\Gamma(R) \in \{\mu, \mu'\}$.

Assume that $\Gamma(R) = \mu$. If $\tilde{R}_{h_3}$ is an egocentric preference such that

$$s_1 \, \sigma(\tilde{R}_{h_3}) \, s_3 \, \sigma(\tilde{R}_{h_3}) \cdots ,$$

then μ' is the only stable matching when the students' preferences are $(R_{-h_3}, \tilde{R}_{h_3})$.

Therefore, student h_3 has incentives to report false preferences when the other students report R_{-h_3}, since

$$\Gamma(R_{-h_3}, \tilde{R}_{h_3}) P_{h_3} \Gamma(R).$$

Incompatibility between stability and strategy-proofness

Theorem 1

There are contexts $(S, H, \succ, q)$ in which no stable and strategy-proof mechanism exists.

Proof (continued). Assume that $\Gamma(R) = \mu'$. If $\tilde{R}_{h_2}$ is an egocentric preference such that

$$s_2 \, \sigma(\tilde{R}_{h_2}) \, s_n \, \sigma(\tilde{R}_{h_2}) \cdots ,$$

then μ is the **only** stable matching when students' preferences are $(R_{-h_2}, \tilde{R}_{h_2})$.

Therefore, the student h_2 has incentives to report false preferences when the other students report R_{-h_2}, since

$$\Gamma(R_{-h_2}, \tilde{R}_{h_2}) P_{h_2} \Gamma(R).$$

Therefore, in any school choice problem compatible with the above-described characteristics, no stable mechanism $\Gamma : \mathcal{R}^{\text{ego}} \to \mathcal{M}$ is strategy-proof. □

Weak externalities vs. absence of externalities

Theorem 2

Given a school choice context $(S, H, \succ, q)$ and a mechanism $\Phi : \mathcal{R}^{\mathrm{std}} \to \mathcal{M}$, define $\Gamma : \mathcal{R}^{\mathrm{ego}} \to \mathcal{M}$ by

$$\Gamma[R] = \Phi[\sigma(R)], \qquad \forall R \in \mathcal{R}^{\mathrm{ego}}.$$

Then, Γ is strategy-proof **if and only if** Φ is strategy-proof and non-bossy.

This result implies that for every school choice context there exists at least one Pareto-efficient, individually rational and strategy-proof mechanism.

When evaluating centralized school allocation mechanisms in the presence of externalities, the efficiency has advantages over the stability.

Incompatibility between stability and strategy-proofness

Suppose $H = \{h_1, h_2, h_3\}$, $S = \{s_1, s_2, s_3\}$ and $q_{s_1} = q_{s_2} = q_{s_3} = 1$. Let $R \in \mathcal{R}^{\text{ego}}$ a profile that induces standar preferences that satisfy the following properties:

$\sigma(R_{h_1})$	$\sigma(R_{h_2})$	$\sigma(R_{h_3})$	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
s_2	s_1	s_1	$\mathbf{h_1}$	h_2	h_2
s_1	s_3	s_2	$\mathbf{h_2}$	$\mathbf{h_3}$	h_3
s_3	s_2	s_3	$\mathbf{h_3}$	$\mathbf{h_1}$	h_1

Further, assume that the preferences of h_2 satisfy that $\mu' \succ_{h_2} \mu$, where

$$\mu = \begin{bmatrix} h_1 & h_2 & h_3 \\ s_1 & s_3 & s_2 \end{bmatrix}, \qquad \mu' = \begin{bmatrix} h_1 & h_2 & h_3 \\ s_2 & s_3 & s_1 \end{bmatrix}.$$

In this context, μ is the only stable matching. However, all students strictly prefer μ'.

Thus, in the presence of externalities, there are school choice contexts in which **there is no** stable and weakly Pareto efficient mechanism.

Ergin-acyclicity

The priority orders $\succ = (\succ_s)_{s \in S}$ and the quota vector $q = (q_s)_{s \in S}$ are Ergin-acyclicity if doesn't exist schools $s_1, s_2 \in S$ and students $h_1, h_2, h_3 \in H$ such that

- $h_1 \succ_{s_1} h_2 \succ_{s_1} h_3$.
- $h_3 \succ_{s_2} h_1$.
- There exist sets $H_{s_1}, H_{s_2} \subseteq H \backslash \{h_1, h_2, h_3\}$, with $|H_{s_1}| = q_{s_1} - 1$ and $|H_{s_2}| = q_{s_2} - 1$, such that $H_{s_1} \subseteq \{h \in H : h \succ_{s_1} h_2\}$ and $H_{s_2} \subseteq \{h \in H : h \succ_{s_2} h_1\}$.

Proposition 1

Given $(S, H, \succ, q)$, the following conditions are equivalent:

(i) $(\succ, q)$ is Ergin-acyclic.

(ii) The mechanism $\mathrm{DA}^{\mathrm{ego}}_{(\succ,q)} : \mathcal{R}^{\mathrm{ego}} \to \mathcal{M}$ is stable and strategy-proof.

Moreover, if $(\succ, q)$ is Ergin-acyclic, then $\mathrm{DA}^{\mathrm{ego}}_{(\succ,q)}$ is Pareto efficient in $\mathcal{R}^{\mathrm{ego}}$.

Existence of stable and strategy-proof mechanisms

Proposition 2

Given $(S, H, \succ, q)$, if there exists a mechanism $\Gamma : \mathcal{R}^{\mathrm{ego}} \to \mathcal{M}$ that is stable and strategy-proof, then it coincides with $\mathrm{DA}^{\mathrm{ego}}_{(\succ,q)}$.

Theorem 3

Given a school choice context $(S, H, \succ, q)$, there exists a stable and strategy-proof mechanism $\Gamma : \mathcal{R}^{\mathrm{ego}} \to \mathcal{M}$ **if and only if** $(\succ, q)$ is Ergin-acyclic.

Resume

	Model without Externalities	Egocentric Preferences
There is a stable and strategy-proof mechanism	✓	×
There is a stable and weakly Pareto efficient mechanism	✓	×
$(\succ, q)$ Ergin-acyclic $\iff$ There is a stable and strategy-proof mechanism	×	✓
There is a stable, strategy-proof, and weakly Pareto efficient mechanism	✓	×
There is a Pareto efficient, individually rational, and strategy-proof mechanism	✓	✓

The effects of externalities on incentives to reveal information may be relevant, even when they play a secondary role in students' preferences for schools.